Chapter 13
A Montessori-Inspired Career in Mathematics Curriculum Development: GeoGebra, Writing-to-Learn, Flipped Learning

Kathy A. Tomlinson

Abstract With an overview of Montessori education, I set the stage for curriculum materials aimed at improving undergraduate mathematics education. I describe four ways to enhance student learning with the dynamical mathematics software GeoGebra: classroom demonstrations, student activities with instructor-created applets, student activities with applets that students create by following podcast instructions, and student-created applets that more advanced students generate independently to solve problems. I discuss two types of writing-to-learn assignments: guided reflection and journaling. I also describe collaborative classroom activities, including associated video lessons that I constructed to implement a flipped or blended learning environment. Connections are made between current mathematics education research findings, Montessori principles and the curriculum materials that I designed. The chapter closes with a reflection on my career path. I discuss my passion for mathematics and social justice, how this led to professional opportunities in mathematics education including a project in the scholarship of teaching and learning, and how my work in mathematics education is useful as I assume leadership as chair of my department.

Keywords Curriculum development • Flipped learning • GeoGebra • Montessori • Writing-to-learn

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MSC Codes
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K.A. Tomlinson (✉)
Department of Mathematics, University of Wisconsin-River Falls,
410 South Third Street, River Falls, WI 54022, USA
e-mail: kathy.tomlinson@uwrf.edu

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13.1 Introduction

In a career that is inspired by Maria Montessori's ideas, I design and implement mathematics curriculum materials that attempt to respond to current mathematics education research. Most of my work seeks to improve instruction in mathematics courses taken by Science, Technology, Engineering and Mathematics (STEM) majors including calculus, differential equations, linear algebra, mathematical modeling, complex variables and discrete mathematics.

As I develop mathematics curriculum I am guided by questions about how students learn and what teaching methods and strategies work for them. How can we help students get a deep conceptual understanding through work with concrete ideas in a way that helps them move to greater abstraction? How can we get students to spend more productive time on task? How can we teach in ways that help students retain knowledge? How can we lower the number of students who withdraw from or fail our classes, while maintaining high learning expectations? How can we help students become engaged with and committed to mathematics?

These questions led me to three forms of curriculum work. The first uses the open source dynamical mathematics software GeoGebra. I have created four types of GeoGebra[1] modules ranging in level of student involvement from the instructor demonstrating in class while students make observations and connections, to students creating their own applets (small computer applications that demonstrate mathematical concepts), making decisions and discoveries along the way. The second focuses on writing-to-learn assignments,[2] encouraging students to reflect and engage with mathematical ideas at many levels. In my third form of curriculum work, I have implemented flipped or blended learning pedagogy, creating collaborative classroom activities supported by video lessons.

13.2 Inspiration from Montessori Mathematics

13.2.1 Principles of Montessori Education

Since Montessori principles have had such a strong influence on the ways I think about teaching and learning, I will outline some key Montessori ideas. While Montessori education is designed for children ages birth through 18, I have found that some Montessori principles translate to the university setting. At the elementary level, Montessori education is characterized by its distinctive classroom environment, teacher role and cognitive goals.

[1] I have created a GeoGebra Book for this chapter: https://www.geogebra.org/book/title/id/RdxKWn2R?doneurl=https%3A%2F%2Fwww.geogebra.org%2Fmaterial%2Fedit%2Fid%2FRdxKWn2R#

[2] Sample writing assignments are available at https://www.uwrf.edu/MATH/SampleMathematicsActivities.cfm

The classroom space is a cross between a cozy living room and a science laboratory. There are open spaces where children spread out their work on rugs, small tables that seat between one and four children, and low shelves where children retrieve beautiful materials that they use in discovery style learning activities. The furniture is arranged to create attractive spaces for each part of the curriculum. Children spend three years in a single room, normally with the same instructor. This enables younger children to benefit from the influence of older children while older children gain leadership experience.

The role of the Montessori teacher is to prepare and organize the learning environment, to provide brief lessons on how to complete learning activities, and above all, to skillfully observe children. Based on these observations, the teacher chooses lessons that capture the child's attention and help each child to make progress at a pace that is appropriate for that child. Using carefully designed hands-on materials, the teacher gives lessons to small groups of children. The children then work autonomously, responsible to practice with the materials over time until mastery is achieved. The teacher serves as a critical link between the child and the prepared learning environment, facilitating the child's construction of his or her own understanding.

Even the cognitive goals in Montessori education are distinctive. They include helping children become self-disciplined, caring, independent, self-motivated, comfortable with error, and able to focus for extended periods. These goals are less tangible than the usual academic content goals and very difficult to measure, especially in a public school setting. Yet giving greater emphasis to these goals often results in higher levels of academic success (Dohrman et al. 2007; Lillard and Else-Quest 2006). One way these goals are attained is through a three-hour uninterrupted work cycle in which children are free to choose what to work on and how much time to devote to it. This promotes problem-solving and concentration by encouraging children to choose challenging work, knowing they will have plenty of time to complete it.

13.2.2 Montessori Principles and College Mathematics

How can these ideas about educating children find relevance in college-level mathematics instruction? While many Montessori practices are specific to the education of children, some of the principles behind the practices are applicable at the college level. Montessori recognizes an important connection between *movement* and *cognition*. Materials are designed to be *self-correcting*. In a college classroom, I have found that hands-on activities using dynamic mathematics software provides students a way to check their work by examining multiple representations. Montessori values *choice* and requires children to *create their own mathematics exercises*. I design mathematics curriculum materials that give students some choices about what mathematical objects to work with and require them to create some of their own exercises. By removing competition and grading systems, Montessori also promotes within the child an *intrinsic motivation* to learn. Inspiring college mathematics students to learn simply because the ideas are so beautiful, important and engaging is one of my greatest challenges. Perhaps the greatest gift of Montessori principles for college level mathematics instruction is its view on the *progression from concrete to abstract*.

As a mathematician, I think about concrete understandings versus abstract understandings in two ways. First, there is the idea of using specific concrete examples to motivate a general abstract principle. We can notice that $23+45$ is even, that $237+841$ is even, and eventually conjecture and then prove, that the sum of any two odd numbers is even. Using pattern recognition to generalize provides one way to progress from the concrete to the abstract.

My second thought about concrete versus abstract relates to the idea of underlying mathematical structure. Mathematicians observe the salient properties of a mathematical object and then generalize to a more abstract version of that object. For example, we notice that Euclidean distance between two points in the Cartesian plane is non-negative, symmetric, zero only when the points are identical, and satisfies a triangle inequality. Based on this observation, we define an abstract metric to be a real-valued function that has these same four properties.

Montessori adds to these understandings of concrete versus abstract in two important ways. First, in the *progression* from concrete to abstract, there are intermediate steps. Montessori mathematics manipulatives are used to guide children gradually from concrete to abstract understanding through a series of small abstractions. Tactile work is associated with the concrete end of the spectrum while purely mental work is on the abstract end. Second, we can teach and learn *a single mathematical concept or process along this progression*, scaffolding student understanding. The mathematics itself is not necessarily getting more abstract, rather the way the student comprehends the mathematics gets progressively more abstract.

In Fig. 13.1 the children are learning to think about place value with a number they chose themselves: 7777. At the most concrete level, they represent 7777 with the "golden beads" (base ten blocks); each golden bead represents one, a bar of ten beads represents 10, a flat of one hundred beads represents 100, a cube of one thousand beads represents 1000. In this representation 7777 is a very tactile concept; there are 7777 beads to touch. Children take the next step in the progression to abstraction with the "stamp game," color-coded tiles with values 1, 10, 100 or 1000 imprinted on them. The stamp game is a more abstract representation of place value than the golden beads because color and numeral, rather than size show the distinction. In another step towards abstraction, the children represent the abstract numeral using color-coded cards whose colors align with the stamp game. The cards with 7000, 700, 70 and 7 are stacked to make 7777. The next material in the progression to abstraction is the "small bead frame," an abacus consisting of four wires each with ten color-coded beads, according to hierarchy. This material is at the abstract end of the spectrum because the numerals imprinted on the stamps are gone and the restriction of ten beads requires the child to do any exchanging between place values immediately. When children work with number operations there are other materials

Fig. 13.1 Progression from concrete to abstract

(not shown in Fig. 13.1) that help students make the transition from concrete understanding of algorithms to abstract paper-and-pencil computations. In all of these representations the child is learning the same concept of place value. The mathematics itself isn't getting any more abstract. However, the child's concept of place value makes a gradual progression on a spectrum from concrete to abstract.

13.3 GeoGebra as a Tool to Improve Conceptual Understanding

GeoGebra, dynamic mathematics open source software, serves as a tool to create Montessori-style activities (think: hands-on, self-correcting) that help students gain abstract understanding through concrete work. Part of the power of GeoGebra is that information may be entered in any one of three ways: symbolically (in the Algebra View using the Input Box), visually (using tools in the Graphics View) or numerically (in the Spreadsheet View). GeoGebra automatically provides the other representations of that same information, cleverly color-coding matching objects in the different representations. Another key aspect is that GeoGebra is dynamic. Once dependent objects have been constructed the user can change one part and the rest of the objects change in a corresponding manner. All of these aspects of GeoGebra work together to provide an experience for students that is hands-on and self-correcting.

13.3.1 Classroom Demonstrations

I began using GeoGebra with classroom demonstrations that I hoped would help students understand the ideas behind the mathematics we were exploring. The GeoGebra software is used to create GeoGebra applets. While there are many such applets available online (GeoGebraTube 2011) for this purpose, I found that writing my own applets gave me better intuition about the power of GeoGebra to support student learning.

One classroom demonstration supports student solutions of an optimization exercise in first-semester calculus. In the exercise students are asked to maximize the area of a rectangle that has its base on the x-axis and its other two vertices above the x-axis and lying on the parabola $y = 8 - x^2$ (Stewart 2008). As with many optimization exercises, the greatest challenge for students is creating the objective function and its domain. Figure13.2 shows one visual result from a GeoGebra applet designed to help students create the objective function, find the domain of the objective function and make sense of their final answer. The applet helps students visualize a concrete sample rectangle. This aids them in understanding the more abstract general rectangle, guiding them next to the discovery that an appropriate expression for the width of the rectangle is $2x$, and then to an appropriate objective function: $A(x) = 2x(8 - x^2)$. As the instructor experiments with the dynamic point (x, y), using the mouse to drag it up and down the parabola, students can discover that an appropriate domain for the objective function is approximately $0 \le x \le 2.8$. The applet also supports the students'

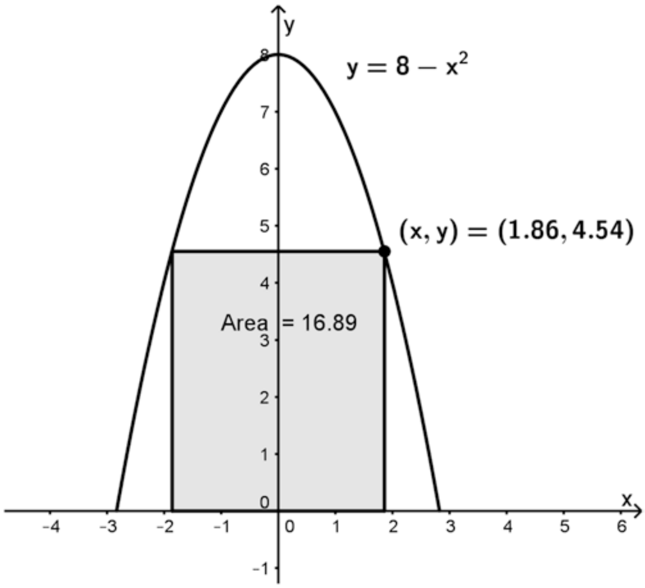

Fig. 13.2 Optimization exercise with GeoGebra

understanding that the upper endpoint of the domain can be found at an x-intercept of the function $y = 8 - x^2$, guiding them to the precise domain, $0 \le x \le \sqrt{8}$. After students have found the critical number of their objective function and tested that their critical number truly maximizes the function, they can make sense of their answer when the instructor experiments with the dynamic point (x, y), observing that the maximum area of the shaded region is approximately 17.4.

This classroom demonstration helps students begin to see how they can create their own objective functions and make sense of their final answers. The dynamic nature of GeoGebra plays an important role in helping students progress from understanding how to think about optimization problems both concretely (a specific rectangle) and more abstractly (a general rectangle).

13.3.2 Student Activities with Instructor-Created Applets

As I created more classroom demonstrations (Tomlinson 2014) and used applets created by others, I realized that it was important for students themselves to interact with the applets. If the research that validates the Montessori principle of movement and cognition (Lillard 2005) is applicable to college students, then students need to get their own hands on the applets. Hence I began to develop *student activities with instructor-created applets* that students access through a learning management system and use to complete exercises both inside and outside of class.

For example, my differential equations students complete exercises outside of class using an applet I developed that illustrates the Euler Approximation Method for first order initial value problems of the form $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$ (see Fig. 13.3). Students complete exercises in which they experiment with different initial value problems, different step sizes and different numbers of steps to get approximate solutions. Their exploration helps them see the connection between the step size and the number of steps. They can also view either an analytic (in the case of an algebraic $f(x, y)$) or a numerical solution, which allows them to make connections between their approximate solution and a more precise solution.

It is fairly easy for students to simply memorize the formulas for Euler's Method: $x_{n+1} = x_n + h, y_{n+1} = y_n + hf(x_n, y_n)$, work exercises from a textbook and correctly solve similar exercises on a test without having even a small clue of what they are doing and what it means. By having students explore Euler's Method exercises with the GeoGebra applet, they begin to make sense of the "$f(x_n, y_n)$" in the formulas and understand that it represents the slope of a line segment. One of the reasons that a unit on Euler's Method is included in a differential equations course is to emphasize that, while many symbolic approaches for solving differential equations are studied, it is important to be able to think about differential equations and their solutions graphically and numerically as well. Thinking in all three modes gives students better intuition about what it means to solve a differential equation. The Euler's Method applet demonstrates these multiple representations very clearly with side-by-side views demonstrating symbolic, visual and numerical representations.

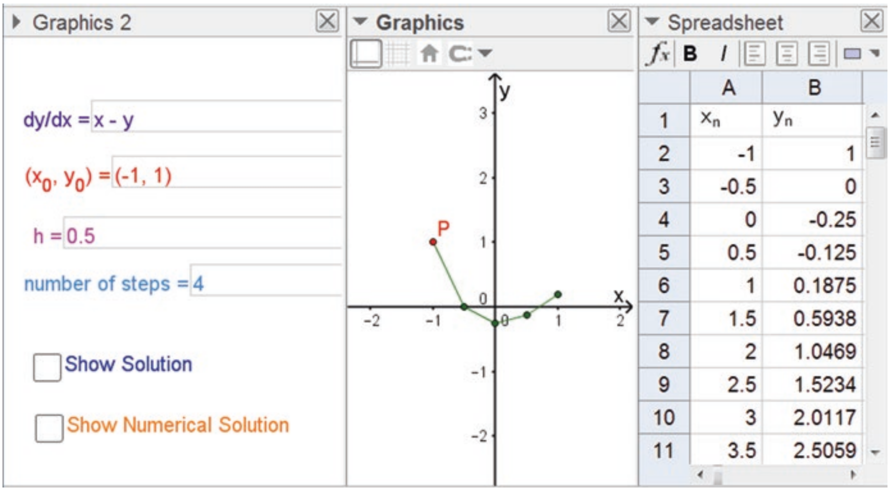

Fig. 13.3 Euler approximation method with GeoGebra

13.3.3 Student-Created Applets Following Detailed Instructions

The third way I use GeoGebra to support student learning is by having *students create their own applets*, outside of class, by following podcast instructions. I provide podcasts instead of live instructions because students can make their applets much more efficiently if they have the ability to pause and re-start my instructions. Once they have created their applets, they bring them to class for exploration to learn mathematics.

Figure 13.4 shows visual output from a student-made applet designed to construct the limit definition of the derivative. In GeoGebra, students graph a function, create a slider, use their slider to draw a dynamic secant line $\overline{PQ}$, and compute the slope of their secant line. Several gains result from having students create the applet themselves. Creating the applet gives students a more concrete understanding of how the coordinates of the points P and Q arise and where the formula for the *slope* of the secant line comes from. It also helps them understand that for a given point P there is a progression of secant lines. So creating an applet helps students gain a deeper understanding of the mathematics. In addition, creating their own applet and assigning colors of their choice gives students a sense of ownership of the knowledge.

In class, students use their sliders to explore the connection between slope of secant line and slope of tangent line. They begin to develop the limit definition of the derivative in a very concrete, visual way. Students explore the results of moving the slider that controls the values of h towards values that are close to 0. This is a tactile version of the abstraction $\lim_{h\to 0}$. By working with the slider students make connections between the slope of a secant line, average rate of change, slope of a

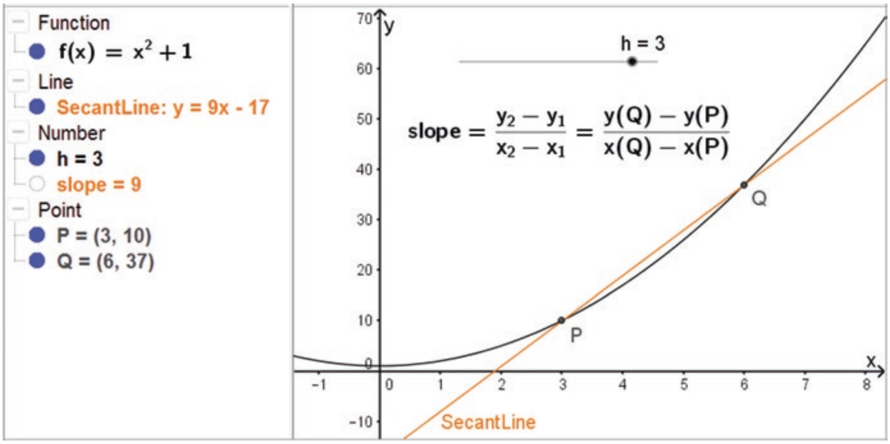

Fig. 13.4 Rate of change with GeoGebra

tangent line and instantaneous rate of change. They come to understand, at a very concrete level, that

$$\text{"slope of tangent line"} = \lim_{h \to 0} \text{"slope of secant line."}$$

The dynamic aspect of GeoGebra next allows students to easily explore the limit definition of the derivative for a variety of functions and points on the functions. Through this exploration students progress from the more concrete formula

$$\text{"slope of secant line"} = \frac{y(Q) - y(P)}{x(Q) - x(P)}$$

to the more abstract formula

$$\text{"slope of secant line"} = \frac{f(x+h) - f(x)}{h}.$$

GeoGebra experiences guide students to combine their rich ideas about the slope of the secant line with their rich ideas about limits to develop the limit definition of the derivative.

13.3.4 Independently-Generated Student Applets

The last type of GeoGebra module I have developed is one for students in more advanced classes to *generate applets independent* of detailed instructions. An example of this type of module is the study of images of lines and circles under

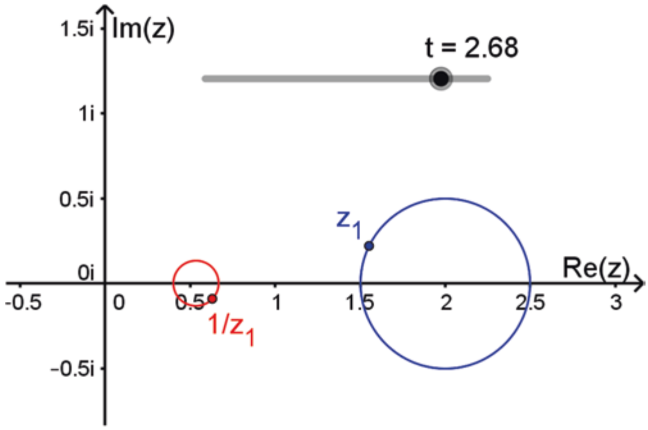

Fig. 13.5 Complex image of circle $\left|z-2\right|=\frac{1}{2}$ under inversion with GeoGebra

complex mappings in a complex variables course. Working with students who are adept at parameterizing lines and circles, I demonstrated how to use GeoGebra to make a conjecture about the image of a line or a circle under the complex mapping $f(z)=z^2$ I followed this by showing the students how to prove those conjectures.

Their out-of-class assignment was to explore the images of circles and lines under the inversion mapping: $f(z)=1/z$ by producing their own applets. Students create an appropriate slider to use as a parameter (see Fig. 13.5) and then use this slider to create a complex number, on a circle or a line (in Fig. 13.5, $z_1=2+0.5e^{it}$). Students use the Trace feature in GeoGebra and the slider to create a line or a circle in the complex plane. Next they define the image of the point z_1. in the GeoGebra Input Box, using the function: $z_2=1/z_1$ Using the Trace feature for the point z_2. students make a conjecture about the image of their line or circle. (In Fig. 13.5 this image is the circle centered at $\frac{8}{15}+0i$ with radius $\frac{2}{15}$) In the assignment students find images of lines and circles under inversion through this progression: (1) a line through the origin of their own choice; (2) circle centered at the origin of their own choice; (3) a particular line that doesn't go through the origin; (4) a particular circle not centered at the origin. By experimenting with these lines and circles students conjecture that the image under inversion of any line or circle is another line or a circle. Working with specific concrete lines and circles, students generalize what happens to any line or circle under inversion.

13.4 Writing-to-Learn Mathematics

When I was an undergraduate majoring in mathematics, my very kind physics instructor attempted to engage me in some casual conversation by asking what we were studying in my advanced calculus class. I was flummoxed. The only answer I could seem to provide was something along the lines of "Section 2.5 exercises 3, 11 and 17." I knew there were some epsilons and deltas involved. I could complete those exercises completely to the satisfaction of my advanced calculus instructor. Yet, there was no way I could give a reasonable response like, "We are learning how to think about continuity in a rigorous, symbolic way. This sharpens the notion of 'arbitrarily close' that we used for limits in calculus class and opens the door to proving theorems about continuous functions." I was woefully inarticulate about what I was learning. Furthermore, I didn't even know what I could have been doing that would help me gain this ability to articulate the mathematics.

As a professor, one lesson I could have taken from this conversation is: "Don't try to make pleasant conversation with your students." Joking aside, for me the real lesson is that if I want my students to learn in a robust way that helps them retain knowledge, I need to find ways to encourage them to articulate what they are learning. By developing writing-to-learn mathematics materials, I help students do this. I am also motivated because these activities have the potential to help students spend more productive time on task. Part of the beauty of writing-to-learn activities is that they can simultaneously help weaker students succeed and provide stronger students with a challenge (Meier and Rishel 1998; Sterret 1992).

The writing-to-learn activities that I use require students to reflect on the mathematics they are learning in ways that facilitate construction of their own understandings. Some writing-to-learn assignments encourage students to verbalize their ideas in dialogue with one another and capture that dialogue on paper. Other assignments involve prompts for inner dialogue resulting in deeper mathematics comprehension. These writing activities help scaffold understanding in the same way that Montessori mathematics materials do for children. While the child in a Montessori classroom is progressing from tactile work (concrete) to mental work (abstract), the college student is progressing from working practice exercises (concrete) to constructing mathematical insights (abstract).

Writing-to-learn assignments differ greatly from proof writing that I explicitly teach in some courses (linear algebra, discrete mathematics, etc.) and from report writing done at the culmination of a semester-long research project in other courses (mathematical modeling, senior capstone, etc.) Descriptions of two types of writing-to-learn exercises I developed follow.

13.4.1 Cooperative Guided Reflection

The first involves projects that I call "cooperative guided reflection" (CGR). In these projects, students work in teams solving textbook exercises and then use a list of prompting questions to guide them in a reflection process. Because CGR is time-consuming, typically students will complete only two CGR projects in a semester. Thus, I choose topics for CGR carefully. A CGR topic should be challenging for students, help students synthesize several ideas, and involve either problem solving or strategizing. Here I will describe how CGR has worked for teaching integration strategies in a second-semester calculus course.

After introducing students to a variety of integration techniques (integration by parts, substitution, etc.) we have a classroom discussion about strategies for deciding which techniques to use. Previously, I would assign about 20 exercises for students to practice with and be done with the topic. To employ CGR, I still assign 20 exercises, but I ask them to complete additional activities in assigned groups of two to four students.

The CGR activities begin with teams of students selecting eight of their 20 integral exercises and creating two integral exercises of their own according to definite guidelines. For example, they must make sure that, broadly speaking, their ten integrals show all of the integration techniques we have studied. They must make sure that they have an integral whose solution requires more than one technique. I provide significant support as students create their own integrals. I give them suggestions that include thinking about the inverse relationship between differentiation and integration, deciding on their technique before they create the integral, and making a variation on an integral from their textbook. They are also expected to check their answers to the integrals they create using technology such as GeoGebra or WolframAlpha®.

The next part of the CGR activities is to reflect on and analyze their ten integrals. They complete a grid in which I list the techniques and they supply a corresponding integral with some verbal explanation to help classify their work. In the last part of the guided reflection, they respond to three prompts asking them to reflect on one integral, one technique and one strategy. The prompts include questions about what they find interesting, how making mistakes helps them learn, and how their decision process works. The writing may be considered informal as students are exploring the way they think about the mathematics in addition to analyzing the mathematics itself. Teamwork promotes student dialog that informs the written reflections.

From the point of view of Bloom's Taxonomy and Webb's Depth of Knowledge (DOK), CGR activities provide higher cognitive demand to students than simply working integration exercises (Hess et al. 2009; Webb 2006). With enough practice, a single integration technique requires low cognitive demand, not much more than recalling and organizing (DOK levels one and two, respectively). Strategizing about which technique to use and using multiple techniques to complete an integral requires higher cognition, as students learn to make decisions and revisions in their integration techniques (DOK level three). CGR activities engage students in the highest cognitive demand, because they extend their thinking, by creating their own integrals and analyzing their integral exercises (DOK level four).

13.4.2 Journaling

The other type of writing-to-learn exercise I have implemented is journaling in the introductory differential equations class. Although the value of mathematics journaling has been written about extensively (Meier and Rishel 1998; Sterret 1992), in my experience it is not a common practice at the college level. Undoubtedly, this is because it can be time-consuming for students and instructors alike. So rather than explain journaling in detail, I will describe how I made this activity manageable both for myself and my students.

In my class, students submit ten journal entries with homework sets. I provide overall guidelines along with some general writing ideas for them. General writing ideas include open-ended instructions to summarize a section of the textbook and to connect differential equations with other disciplines. In the guidelines, I describe the purpose of their journaling: to learn by exploring, organizing and synthesizing mathematical ideas. In addition to such general writing ideas, with each homework set I provide two or three content-specific prompts from which students can choose (Farlow 1994). Although some instructors have used mathematics journals to explore the affective realm, I emphasize that their journal is not a place to discuss course mechanics or exam results, that students are expected to confer with me about such concerns.

Scoring for the journals is based on complete, thoughtful entries. Mostly students get full credit, saving instructor time. I take time to provide positive feedback that emphasizes students' best ideas. I also provide corrections for misconceptions or mistakes. When it is clear that students have put thought into their entries, I do not deduct points for such errors.

Both CGR and journaling reveal student thinking and confusion that can become prompts for classroom discourse. The first time I used CGR with the Integration Strategy topic, I had no idea that students conflated integration *technique* with integration *strategy*. Groups of students wrote that *technique* and *strategy* were two words for the same thing. Our subsequent in-class conversation helped them begin to distinguish between when they were using a technique and when they were making a decision about what technique to use next. I often share some of the best student writing with the entire class by displaying it on a document camera. This provides students with models for how to reflect on mathematics and how to articulate their thoughts. One of the most illuminating student journal entries stated, "I can do all of the assigned exercises, but I don't really understand it well enough to journal about it yet." In the absence of the journaling activity, students may equate working routine homework exercises with truly understanding mathematics. This student realized that he needed to do some more thinking, reflecting, or talking to make complete sense of what we were learning.

13.5 Flipped or Blended Learning

13.5.1 Principles and Goals of Blended Learning

Some of my most recent curriculum work involves implementing *flipped* or *blended* learning, as a way to improve *interactive engagement* during class time. *Interactive engagement* teaching methods involve activities that yield immediate feedback through discussion with peers or instructors (Epstein 2013). Primarily, the term *flipped* refers to a flip between learning from lectures in class and then practicing at home to learning from lectures at home and practicing in class with instructor support. In my classes, this means using technology (video lessons) so that students see ideas at home and then work on exercises in groups in class. A second interpretation of the term *flipped* is that it is a flip between the classroom activities being centered on the teacher to being centered on the student activity. The term *blended* means that there is a mix between a flipped classroom and the more traditional approach. For me, blended learning always involves some practice exercises at home.

There are many pointers guiding us in the direction of interactive engagement teaching methods, from Montessori's focus on student-centered learning to work by science colleagues to incorporate more active learning in their classrooms (Freeman et al. 2014). Research on the Calculus Concept Inventory (CCI) is especially compelling. The CCI is a way to measure students' conceptual (but not necessarily procedural) understanding of calculus. Researchers found that students in US colleges did very poorly on CCI and that none of the following had an effect on CCI score: class size, instructor experience, time spent in class, student preparation at entrance. However, *interactive engagement* teaching methods did improve student performance on CCI (Epstein 2013). A national study of calculus instruction also points toward the efficacy of active learning (Bressoud et al. 2015).

Thinking about using flipped learning to help students who were not succeeding in my classes led me to Bergman and Sams' (2012) delineation of three types of students who do poorly in school. There are students whose time is over-extended; many of my students work more than 20 h per week, while taking a full load of challenging coursework. There are students who have an insufficient background. (This is my personal favorite excuse for students doing poorly in my classes, since it takes the onus off of me.) Helping students who need to fill in missing gaps is a significant part of my teaching. The third kind of student Bergman and Sams identified as "playing school." These students come to class, but don't want to learn, aren't trying to learn, and are instead really just trying to figure out how to get a certain grade, by doing the least amount of work possible. It never occurs to them that learning is in their own best interest. Bergmann and Sams argued that they are able to reach all three types of students through flipped learning. As I design materials to implement flipped learning, I keep these three kinds of students in mind along with the students who have great success in more traditional college learning environments.

When I began to implement flipped learning I had already been creating online video content (accessible through a learning management system), in the form of annotated notes, to support my students as they worked on homework exercises. Students appreciated hearing my voice helping them work an assigned exercise, emphasizing the ways I wanted them to think about various aspects of the work. My strongest students used them occasionally; I could see real gains for students who were underprepared. But students who were very busy or who were "playing school" were not benefiting.

13.5.2 Blended Learning in Calculus I

With flipped or blended learning, some of the lessons are provided to students before class. This frees up class time to support students as they interact with the material, helping diverse learners. This is how the flip worked for me in a first-semester calculus unit on areas. Students were assigned four 5-min podcasts to watch before class. In the video lessons, I explained the general ideas and demonstrated two examples. Students were expected to take notes and be prepared to show them to me at the beginning of class.

When students came to class I distributed a packet of exercises consisting of the same examples I had worked on the chalkboard when teaching this unit in a more traditional format. Students worked on the packet with each other using their notes from the podcasts. Students who had not watched the podcasts (or not taken notes) moved next to a classmate with notes. This arrangement worked well because most students had notes (knowing that there were points associated with them) but those who didn't still had a way to be fully engaged. I began to circulate around the room, checking podcast notes and talking to students. Every few minutes I took a break to write down some solutions, projected onto the document camera. This gave students a way to check their work and also kept them on task. A few times, I paused the class work briefly to direct a whole group discussion, addressing an idea that had arisen. As I circulated through the room, I answered individual questions about both the video lessons and the packet of exercises, many of which I would have been unlikely to hear in my more traditional format. I had a personal interaction with each one of my 32 students that day.

There was one student in my class who I knew led a very busy life and who had also missed some class days because of illness. When I checked in with her during class that day, she said, "I'm doing fine with areas, but I am still having problems with integration by substitution," the topic we had been working with the previous week. Normally, I might have asked her to come to my office hours (which was unlikely to actually happen because of her schedule). In the flipped format, I could see that everyone was on task with areas, so I had time to address her questions immediately.

For me, flipped or blended learning is a way to lower the number of students who do poorly in my classes, while maintaining high learning expectations. It is a way to help all students become engaged with and committed to mathematics. There is one major drawback: the amount of instructor time required to prepare video lessons and in-class activities. I did not use materials that others have posted on the internet, because I believe I can create a video lesson in less time than I can find a suitable version online and because of recommendations about students' need to connect with their instructor (Moore et al. 2014). With flipped or blended learning, I have more meaningful interaction with my students, the most pleasurable part of teaching.

13.6 Career Trajectory

My career has been fueled by passions for mathematics and for social justice. I started as a researcher in partial differential equations, investigating questions about the heat equation with space variables that are complex. While I have had a lifelong interest in political, social and economic equality for all, I didn't originally see this as part of my career. However, when my campus was looking for someone to lead our Women's Studies Program, I saw a way to realize my passion for social justice and I took the opportunity to take my career on a brief excursion. As Director of Women's Studies, I taught women's studies courses and coordinated women's studies programming with faculty from a wide array of disciplines. This work led me to the epiphany that mathematics education is a social justice issue. By creating high-quality mathematics classrooms that spark curiosity and foster long-term interest in mathematics, we are helping to create equal access to our economy (Halpern et al. 2007).

The next detour in my career path was motivated by the birth of my children, leading me to a study of Maria Montessori's idea that the most effective education is supported by materials and activities that are hands-on, self-directed, self-correcting and self-chosen. I became an advocate for Montessori education, presenting to community groups and serving on school committees. Through grant-funded work with the College of Education on my campus and volunteering in local schools, I work to bring Montessori mathematics into mainstream classrooms. Eventually, I began using Montessori's ideas in my own college mathematics classrooms.

The opportunity to do mathematics education research presented itself when I participated in a regional scholarship of teaching and learning (SoTL) in mathematics workshop. I completed a project addressing the question of how a cooperative guided reflection (CGR) activity in first-semester calculus improved problem-solving skills by doing a literature review, a quasi-experiment, student surveys about problem-solving, and an analysis of student work. I found that there was a positive impact, qualitatively, on students' mathematical belief systems, as well as quantitatively, on students' ability to solve optimization problems (Tomlinson 2008). This gave me impetus to continue experimenting with CGR.

Another important aspect of my career has been coaching 19 successful mathematical modeling teams. It is a joy to help these students develop skills in mathematics, internet research, teamwork, mathematical technology, and technical writing. Working closely with these students informs my thinking about how to create instructional materials for students at their level.

When GeoGebra, became available, it was a perfect fit, providing another way to implement Montessori principles. The fact that GeoGebra is open source was especially attractive to me because it means that everyone has access. As we are learning more about the importance of interactive engagement classrooms and flipped learning, I am creating curriculum materials to implement these pedagogies as well.

Regular exchange of ideas with colleagues has been key to my success and my continued energy for creating instructional materials. This takes many forms: informal comparison of topic treatment with colleagues in my department who teach the same courses I am teaching, formal discussions with faculty in science departments about how mathematics courses support their work, participation in grant-funded work with colleagues in Teacher Education, and discussions with colleagues outside of my campus at conferences.

My work has been well received on my campus. Any work that improves student engagement usually results in improved retention and recruitment—priorities at most colleges and universities. I have given faculty development presentations on GeoGebra for adjunct and regular faculty, and served as the contact person for people with technology and pedagogy questions about this software. This has been appreciated by faculty and administrators alike. I have accepted invitations to lead GeoGebra workshops for faculty at a local high school and a regional two-year college. Survey responses of students on my work with flipped pedagogy are very positive.

My career has gradually shifted from esoteric, but definitive, questions about partial differential equations to broad-reaching, but nebulous questions about better ways to teach mathematics. It is a comfort to start with a mathematical premise, logically arrive at a conclusion and know that this work is entirely repeatable. On the other hand, it is exciting to create materials that help at least some students become committed and engaged in mathematics, even if we cannot always be certain that the same materials will work for a different instructor or a different set of students.

In the next phase of my career, I am learning how to provide leadership to an academic department of nine tenure-track faculty and 11 other instructional staff members as I begin to serve as chair. While I continue my work developing instructional materials, I am taking on a greater role promoting high-impact mathematics education practices in my department. I believe that this focus will, over time, strengthen my department by making our graduates more employable and attracting more students to our programs. I have started this emphasis by using our curriculum review process as a way to share research results from mathematics education among my faculty. I am learning that through internal grants and personnel processes, I have new opportunities to encourage my faculty to pursue work that improves mathematics education in all of our classrooms.

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